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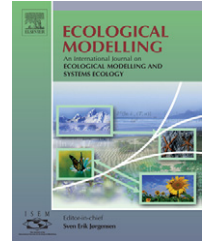
Short communication

Incorporating output variance in local sensitivity analysis for stochastic models

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ABSTRACT

The output of stochastic models is a distribution of values, rather than a single value such as in deterministic models. Local sensitivity analyses of such models typically ignore the higher moments of the output distribution and instead use the distribution mean to represent model output. This might be simplistic, since the shape of the distribution might also be sensitive to changes in model parameters. Here, we construct a simple sensitivity index that captures also the shape of the output distribution, by incorporating its variance in addition to its mean. To evaluate its performance, we reconstructed an existing stochastic individual-based model for mosquitofish (*Gambusia holbrooki*) population. We compared the performance of the new sensitivity index to the standard sensitivity index ($\partial Y / \partial P$) that was calculated using the mean of the output distribution, by ranking model parameters according to their impact on the output. Sensitivity analyses using both methods identified different parameters as the most influential on model output, and rankings were inconsistent between methods regardless of the number of simulations used for generating the output distributions. It is shown that the new index indeed captured better the effect of parameters on model output since it accounted for the variance of the output distribution.

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1. Background

Sensitivity analysis (SA) is a step in the modeling process aimed to rank model parameters, initial values of state variables, sub-models, or even processes (Brugnach, 2005) according to their impacts on model results (Jorgensen, 1994; Grimm and Railsback, 2005). A parameter that the model is sensitive to is one that minor changes in its value would result in major changes in model output or inference. SA has an important role in the modeling process, since in many cases, parameter values are uncertain (Cariboni et al., 2007) due to the complexity of natural systems or the ways that they were obtained. In some cases, model parameters are estimated by empirical observations or field experiments, where there is at least an estimate of the degree of uncertainty around the estimate (Fieberg and Jenkins, 2005). In many other cases, initial parameter values are derived from expert opinion, which is typically characterized by large uncertainty (Haimes, 2004; Ray and Burgman, 2006). When high uncertainty coincides with high sensitivity in model parameters, the reliability of model predictions may be very low.

SA is performed in order to answer one or more practical needs arising in the modeling process (Cariboni et al., 2007). The two most common goals are research prioritization and

reducing complexity. In research prioritization, researchers need to decide which model parameters need to be better estimated (typically those with larger impact on the model). Reducing model complexity can be achieved by filtering out parameters or processes that have minor effects on model outcome (Lawrie and Hearne, 2007).

In general, SA is performed by modifying the values of model parameters by various quantities, re-running the model, and computing the changes in model output relative to its output with initial parameter values ("reference run"). At its most simple form, SA is performed by changing parameter values one at a time, while fixing all other parameter values. This process, known as local SA, detects the net effects of single parameters. Global SA is a group of more robust techniques, altering groups of parameters simultaneously, in order to detect the effects of interactions between different parameters (Fieberg and Jenkins, 2005; Ginot et al., 2006; Cariboni et al., 2007; Chu et al., 2007). In local SA, the sensitivity of a model to a single parameter P_i is calculated as (Jorgensen, 1994; Caswell, 2001; Cariboni et al., 2007):

$$S_a = \frac{\partial Y}{\partial P_i} \tag{1}$$

where S_a is the sensitivity index of the state variable Y (representing model output) to the parameter P_i. In practice, the partial derivatives in Eq. (1) are calculated as the differences between original (reference) and new parameters and state variables, in incremental ratios.

In deterministic models, the outcome of a specific set of parameters is essentially the same for identical starting conditions, therefore ∂Y is null. In contrast, in stochastic models the output varies between simulations, even when parameters and starting conditions are identical, due to variability introduced by stochastic model terms. In such cases, a single simulation for assessing the sensitivity of the model to a specific parameter is insufficient, and a set of simulations is needed in order to generate an entire distribution of model outputs. Thus, the computation of sensitivity in stochastic models involves the comparison of two distributions rather than two single values.

The common approach for comparing a pair of model outcomes is to calculate the mean values of the resulting output distributions and use them for the computation of the numerator terms in Eq. (1) (Brugnach, 2005; Ginot et al., 2006). This solution might be sufficient only in cases where the output variance is insensitive to parameter values. Since the distribution of outputs is unknown before running model simulations, using the averages while ignoring the shape of the distribution might lead to inaccurate results, especially when differences between the sensitivities to different parameters are slight.

A more thorough approach for SA of stochastic models would also account for the shape of the distribution. Different output distributions might have the same means, but very different variances, therefore the sensitivity score of the corresponding parameters cannot be assumed equal, since this essentially means that the model reacts differently to changes in different parameters. A potentially better approach would be to account for the variance of the distribution in addition to its mean, thus incorporating the second moment of the dis-

tribution in addition to its first. In global SA, this is sometimes done in ANOVA-related methods (Chaloupka, 2002; Ginot et al., 2006), and other methods (Campolongo et al., 2007). We suggest that the t-statistic (Underwood, 1997) should be used for this task. The t-statistic for a pair of distributions is denoted by:

$$t = \frac{\bar{Y}_{alt} - \bar{Y}_{ref}}{\sqrt{(s_{alt}^2 + s_{ref}^2)/n}} \tag{2}$$

where t is Student's t-statistic of two equal sized samples, $\bar{Y}_{alt}$ is the mean of the state variable distribution generated from an altered parameter, $\bar{Y}_{ref}$ is the mean of the state variable distribution generated from the original (reference) parameter, n is the sample size, and s_{alt} and s_{ref} are the standard deviations of those distributions. The larger the absolute value of t is, the greater the difference between the two distributions.

Since t accounts for the shape of the two distributions in addition to their means, it can be assumed that it will display the true difference between them better than the mean values only. Assuming discrete parameter values and replacing ∂Y with t_y, a local sensitivity index around a reference parameter value that is based on t can be formed based on Eqs. (1) and (2):

$$S_t = \frac{t_y}{(p_{alt} - p_{ref})/p_{ref}} = \frac{|\bar{Y}_{alt} - \bar{Y}_{ref}|}{(p_{alt} - p_{ref})\sqrt{(s_{alt}^2 + s_{ref}^2)/n}} p_{ref} \tag{3}$$

where p_{alt} and p_{ref} are the new and the reference parameter values, respectively, t_y is the t-statistic of the new and reference output distributions, and S_t is the t-based local sensitivity index. The denominator is divided by p_{ref} in order to normalize the values of different parameters, to allow comparisons. This approach is somewhat similar to using analysis of variance (ANOVA) for assessing the differences between the effects of more than two parameters (Chaloupka, 2002; Ginot et al., 2006). But, since t is used here only as a measure of parametric distance, and no actual statistical test is performed (in contrast to the F-test in ANOVA), there is no need to satisfy the requirements of normality and variance equality that are necessary for application of ANOVA.

Another aspect of SA for stochastic models that is generally overlooked is the effect of number of simulations on the results. Since statistical moments used for stochastic SA are only estimates (as the true distribution is unknown), their values depend on the number of simulations that were used to generate the output distribution. This essentially leads to difficulties in interpreting SA results, because different numbers of simulations may yield different parameter rankings, with increased accuracy achieved only when the number of simulations is large. Therefore, it is desirable that the SA will be performed on a distribution that is based on a sample size that is large enough, so that the inherent stochasticity of the model will not affect SA results.

The objective of this research is to introduce, explore, and validate the application of S_t as a better measure of local sensitivity in stochastic models in small sample sizes. In order to do so, we conducted a large number of simulations using an existing stochastic model of fish population (Ginot et al., 2006).

We then compared the performance of the standard sensitivity index, S_a, to that of the new index, S_t, in a local SA, at varying numbers of simulations.

2. Methods

2.1. Model description

The mosquitofish stochastic individual-based model developed by Ginot et al. (2006) was reconstructed for the purpose of this study (Fig. 1). The model simulates the population growth of mosquitofish (*Gambusia holbrooki*) in experimental tanks. The model has eight parameters (Table 1) and one state variable—population size. The model includes three kinds of individuals: juveniles, adult males, and adult females. The time step of the model is one day. The processes affecting the life of each individual depend on its kind, and are: (1) survival; (2) age growth; (3) length growth; (4) puberty for juveniles; and (5) offspring production for females.
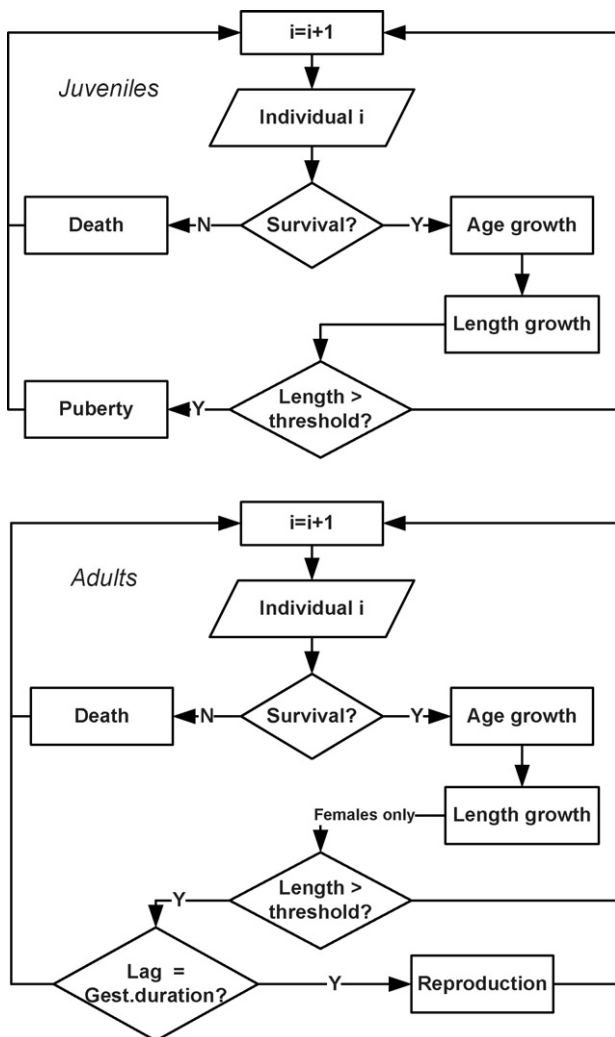

Fig. 1 – Schematic representation of the mosquitofish model (Ginot et al., 2006) for juveniles (top) and adults (bottom). Notice that males undergo only part of the process.

Table 1 – Parameter values for the mosquitofish model, adopted from Ginot et al. (2006)	
Parameter [unit]	Reference value
Growth rate [day^{-1}]	0.9548
Maximum length [mm]	60
Puberty threshold [mm]	15
Adult survival [%, day^{-1}]	0.9965
Juvenile survival [%, day^{-1}]	0.9607
Gestation duration [days]	30
Fecundity threshold [mm]	24
Fecundity [juveniles mm^{-1}]	2

The survival of an individual is tested each time step according to its survival probability, which is a parameter. The increase in length between two time steps is defined by the following function:

$$\text{Length}(i + 1) = \text{Length}(i) + a \left(1 - \frac{\text{Length}(i)}{K} \right) \qquad (4)$$

where Length(i) [mm] is fish length at day i, a [day^{-1}] is the growth rate parameter, and K [mm] is the maximum length parameter. Therefore, growth rate decreases linearly according to length.

Puberty and breeding are length dependent. When the length of a juvenile exceeds the puberty threshold parameter, it becomes an adult, either male or female according to a 1/1 sex ratio. Similarly, when the length of a female exceeds the fecundity threshold parameter, and additionally, a certain time lag has passed since its last parturition (denoted by the gestation duration parameter), it reproduces offspring. The number of offspring is determined by

$$N_{\text{offspring}} = \text{Fecundity} \left(\text{Length}(i) - F_{\text{thresh}} \right) \qquad (5)$$

where Fecundity [juveniles/mm] is a parameter that represents the number of offspring created per unit length, and F_{thresh} [mm] is the fecundity threshold parameter. The offspring are assigned a random length between 7 and 8 mm. Model simulations begin with a population of 10 females, with lengths varying randomly between 22 and 26 mm, and ages between 0 and 30 days. Additional stochasticity in individual specific parameters in the original model (Ginot et al., 2006) was not incorporated here, since we wanted to account only for the stochasticity introduced by model structure. Simulation periods were 120 days.

2.2. Sensitivity analysis

Model simulations were repeated 50, 100, 500, 1000, and 5000 times to assess the effects of the number of simulations on the SA. Model parameters were altered one at a time by multiplying their reference value by 1.15 (i.e. +15%). The state variable of interest was the population size at day 120. The average and standard deviation of each state variable distribution were calculated for each set of repetitions. Two sensitivity indices, S_a and S_t, were calculated for all parameters. Model parameters were ranked according to their sensitivity index value. Param-

eter rankings generated by the two sensitivity indices were compared, in order to identify potential differences between these two approaches. Additionally, parameter rankings generated in different numbers of simulations were compared within each index, in order to identify the number of simulations from which additional simulations will not affect the parameter rankings.

3. Results

Model simulations over 120 days with the starting conditions described above resulted in an exponential population growth of mosquitofish (Fig. 2). Each simulation result differed from the others due to the stochasticity of the model. Population size at year 120 was sensitive to changes in all parameter values at various levels, and the sensitivity was affected by the number of simulations (Fig. 3). In all cases, S_a and S_t generated inconsistent parameter rankings.

According to S_a, the model was most sensitive to growth rate, followed by fecundity threshold. In contrast, according to S_t, the model was most sensitive to fecundity threshold, followed by growth rate. Therefore, the two methods differ in the rankings of the two most influential parameters (Fig. 4). This is nicely explained by the features of these methods: modifying growth rate affects the difference between means (of the reference and modified distributions, respectively) more than modifying fecundity threshold, making growth rate the most influential parameter in S_a. However, S_a overlooks an important trait: modifying growth rate increases output variance (and even leads to some overlap between reference and modified distributions), while fecundity threshold decreases output variance (Fig. 4), making fecundity threshold the most influential parameter in S_t.

In both metrics, the number of simulations affected parameter rankings, for the parameters that had small impact on model results (Fig. 3). For S_t, the ranks of fecundity threshold, growth rate, and juvenile survival remained constant

Fig. 3 – Local sensitivity values of the population size at day 120 at different numbers of simulations. Bars represent S_a (left column) and S_t (right column). Each row represents a different number of simulations. Parameters are numbered in the following order: (1) growth rate, (2) maximum length, (3) puberty threshold, (4) adult survival, (5) juvenile survival, (6) gestation duration, (7) fecundity threshold, and (8) fecundity.

regardless of the number of simulations (ranked 1st, 2nd, and 8th, respectively). For S_a, the ranks of growth rate, fecundity threshold, and maximum length remained constant regardless of the number of simulations (ranked 1st, 2nd, and 3rd, respectively). Increasing the number of simulations from 1000 to 5000 had no effect on the rankings in both metrics.

4. Discussion

Sensitivity analysis of stochastic models that is based on average values of the output distributions may not be robust, since it ignores changes in the shape of the output distribution. Sensitivity indices that account for statistical moments other than the average are potentially more effective in capturing these stochastic effects, thus they may be considered conceptually better than using the average solely. This was shown here through a rather simple stochastic ecological model, the mosquitofish IBM by Ginot et al. (2006). The sensitivity index suggested here, S_t, although simple, offered more information on the sensitivity of model outputs to input parameters than
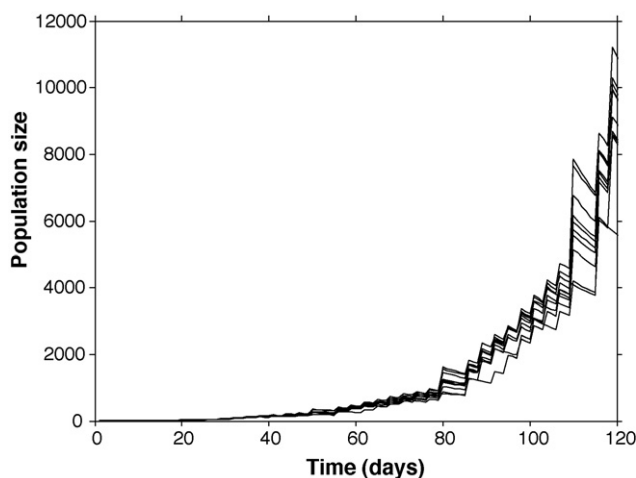

Fig. 2 – Changes in mosquitofish population size during a period of 120 days in 10 different repetitions. Each line represents a different repetition with identical starting conditions, in order to represent the stochasticity of the model.
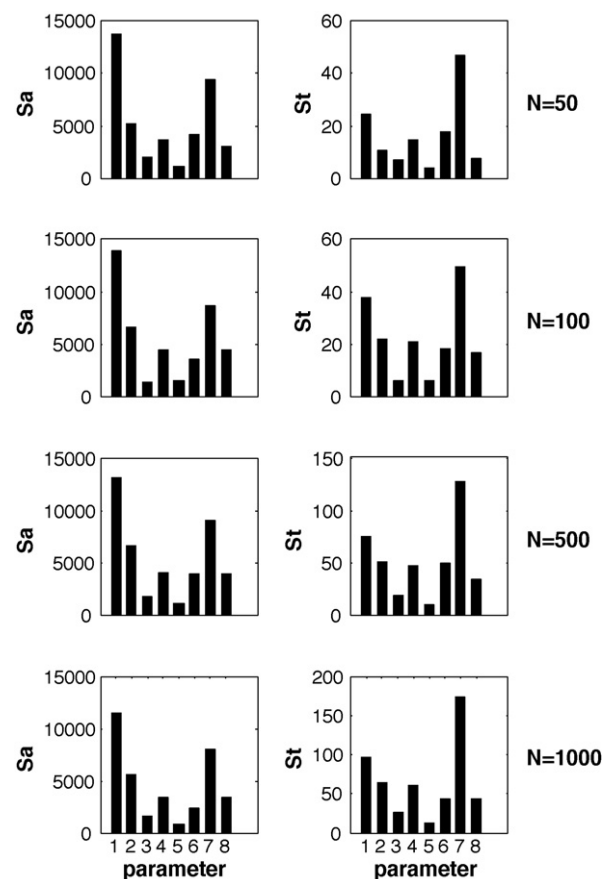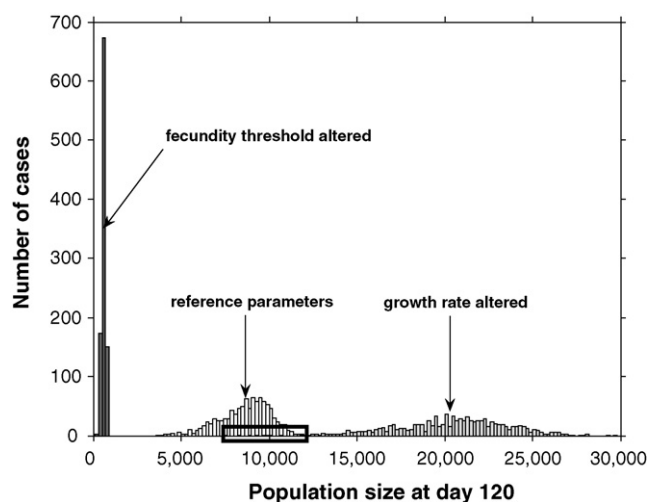
Fig. 4 – Output distributions (population size at day 120) of three groups of model simulations based on 1000 repetitions: using reference parameters (hollow bars, middle), following alternation of the growth rate parameter by +15% (light gray bars, right), and following alternation of the fecundity threshold parameter by +15% (dark gray bars, left). Arrows indicate the locations of the distribution means. The overlap area between the reference output distribution and the output distribution based on altered growth rate is marked by a black rectangle.

S_a did. The difference in parameter rankings between the two indices was evident, and occurred at all levels of change and simulation numbers.

The number of simulations affected both sensitivity indices. This is because both indices are based on estimates of statistical moments of the output distributions, and these estimates are essentially better when the number of simulations increases (Underwood, 1997). The effect of number of simulations is more pronounced when the differences between sensitivities to output parameters are small, since in these cases the inherent stochasticity of the model might obscure the subtle differences between parameter effects. However, at least in the mosquitofish model, the three most influential parameters were insensitive to the number of simulations, since their impact on model output was strong.

The effectiveness of S_t is expected to be pronounced in models that have at least a moderate amount of inherent stochasticity (i.e. there are different sources of stochasticity, either by parameters or by model structure). In deterministic models, and in models with limited stochasticity, S_t converges with S_a, since the output variance is non-existent (deterministic models), correlated with the output means (linear models),

or unaffected by the stochastic terms of the model (models with a single source of stochasticity).

Accounting for higher statistical moments other than the mean is desirable in stochastic models, since it adds additional insight to the impact of parameters on model behavior. The S_t index introduced here offers a simple and straightforward solution towards this means.

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